X. *On Plane Water-Lines in two Dimensions.* By WILLIAM JOHN MACQUORN RANKINE, C.E., LL.D., F.R.SS. L. & E., Associate of the Institution of Naval Architects, &c.

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SECTION I.—*Introduction, and Summary of known Principles.*

(1.) *Plane Water-Lines in two Dimensions defined.*—By the term "Plane Water-Line in two Dimensions" is meant a curve which a particle of liquid describes in flowing past a solid body, when such flow takes place in plane layers of uniform thickness. Such curves are suitable in practice for the water-lines of a ship, in those cases in which the vertical displacements of the particles of water are small compared with the dimensions of the ship; for in such cases the assumption that the flow takes place in plane layers of uniform thickness, though not absolutely true, is sufficiently near the truth for practical purposes, so far as the determination of good forms of water-line is concerned. As water-line curves have at present no single word to designate them in mathematical language, it is proposed, as a convenient and significant term, to call them *Neoïds* (from νηὸς, the Ionic genitive of ναῦς).

(2.) *General Principles of the Flow of a Liquid past a Solid.*—The most complete exposition yet published, so far as I know, of the principles of the flow of a liquid past a solid, is contained in Professor STOKES's paper "On the Steady Motion of an Incompressible Fluid," published in the Transactions of the Cambridge Philosophical Society for 1842. So far as those principles will be referred to in the present paper, they may be summed up as follows.

When a liquid mass of indefinite extent flows past a solid body in such a manner that as the distance from the solid body in any direction increases without limit, the motion of the liquid particles approaches continually to uniformity in velocity and direction, the *condition of perfect fluidity* requires that the three components u, v, w of the velocity of a liquid particle should be the three differential coefficients of one function of the coordinates (ϕ); viz.

$$u=\frac{d\phi}{dx};\ v=\frac{d\phi}{dy};\ w=\frac{d\phi}{dz}; \quad \dots\dots\dots\dots \quad (1)$$

and the *condition of constant density* requires that the said function should fulfil the following condition,

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0. \quad \dots\dots\dots\dots \quad (2)$$

By giving to the function ϕ a series of different constant values, a series of surfaces are represented, to which each water-line curve is an orthogonal trajectory, so that if U=

constant be the equation of a series of surfaces each containing a continuous series of water-line curves (and one of which surfaces must be that of the solid body), the function U must satisfy the following condition,

$$\frac{dU}{dx}\cdot\frac{d\phi}{dx}+\frac{dU}{dy}\cdot\frac{d\phi}{dy}+\frac{dU}{dz}\cdot\frac{d\phi}{dz}=0; \quad \ldots\ldots\ldots \quad (3)$$

or if ds' be an elementary arc of a water-line curve, and x', y', z' its coordinates, the following conditions must be satisfied,

$$\frac{dx'}{ds'}:\frac{dy'}{ds'}:\frac{dz'}{ds'}::\frac{d\phi}{dx}:\frac{d\phi}{dy}:\frac{d\phi}{dz}; \quad \ldots\ldots\ldots \quad (4)$$

and these are the most general expressions of the geometrical properties of water-line curves in three dimensions.

When the inquiry is restricted to motion in two dimensions only, x and y, the terms containing dz and dz' disappear from the preceding equations; and it also becomes possible to express the same conditions by means of equations of a kind which are more convenient for the purposes of the present investigation, and which are as follows. Conceive the plane layer of liquid under consideration of thickness unity, to be divided into a series of elementary streams by a series of water-line curves, one of which must be the outline of the solid body; let U= constant be the equation of any one of those curves, U being a function of such a nature that dU is the volume of liquid which flows in a second along a given elementary stream; then the components of the velocity of a particle of liquid are

$$u=\frac{dU}{dy};\ v=-\frac{dU}{dx}; \quad \ldots\ldots\ldots \quad (5)$$

the condition of continuity is satisfied; and the condition of perfect fluidity requires that the function U should fulfil the following equation,

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}=0. \quad \ldots\ldots\ldots \quad (6)$$

(When the motion of the liquid is not subject to the condition of being uniform in velocity and direction at an infinite distance in every direction from the solid, it is sufficient that

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}=\text{function of U};$$

but cases of that kind do not occur in the present paper*.)

(3.) *Notation.*—It is purely a question of convenience whether the infinitely distant particles of the fluid are to be regarded as fixed and the solid as moving uniformly, or

* Professor WILLIAM THOMSON, in 1858, completed an investigation of the motion of a solid through a perfect liquid, so as to obtain expressions for the motion of the solid itself, involving twenty-one constants depending on the figure and mass of the solid and the density of the liquid; but as that investigation, though on the eve of publication, has not yet been published, I shall not here refer to it further.

the solid as fixed, and the infinitely distant particles of the fluid as moving uniformly with an equal speed in the contrary direction. Throughout the present paper, the solid will be supposed to move along the axis of x; so that v will represent the transverse component of the velocity of a particle of liquid on either supposition. The longitudinal component of the velocity of a liquid particle *relatively to the solid* will be denoted by u; and when that particle is at an infinite distance from the solid, by c; so that when the infinitely distant part of the liquid is regarded as fixed, the solid is to be conceived as moving with the velocity $-c$; and the longitudinal component of the velocity of a liquid particle relatively to the indefinitely distant part of the liquid will be denoted by $u-c$.

It is convenient to regard the function U as equivalent to an expression of the following kind,

$$\mathrm{U}=bc, \quad \ldots \ldots \ldots \ldots \quad (7)$$

c being the uniform velocity of flow at an infinite distance, and b what the value of y would be for the water-line under consideration if the solid were removed; in which case that line would become a straight line parallel to the axis of x. This enables us to substitute for equations (5) and (6) the following, in which *proportionate* velocities only are considered:—

$$\frac{u}{c}=\frac{db}{dy}; \quad \frac{v}{c}=-\frac{db}{dx}; \quad \ldots \ldots \ldots \quad (8)$$

$$\frac{d^2b}{dy^2}+\frac{d^2b}{dx^2}=0. \quad \ldots \ldots \ldots \ldots \quad (9)$$

(4.) *General Characteristics of Water-Line Functions.*—Since at an infinite distance from the solid body we have $u=c$, $v=0$, it follows that, if the origin of coordinates be taken in or near the solid body, b must be a function of such a kind that, when either $x=\infty$, or $y=\infty$,

$$b=y.$$

Hence in a great number of cases that function is of the form

$$b=y+\mathrm{F}(x, y); \quad \ldots \ldots \ldots \quad (10)$$

where F is a function which either vanishes or becomes constant when x or y increases indefinitely.

It is plain that when the function b takes this form, the term F is the function for the motions of the liquid particles *relatively to still water*; that is to say,

$$\frac{u-c}{c}=\frac{db}{dy}-1=\frac{d\mathrm{F}}{dy}; \quad \frac{v}{c}=-\frac{db}{dx}=-\frac{d\mathrm{F}}{dx}, \quad \ldots \ldots \quad (11)$$

and also that the term F fulfils the equation

$$\frac{d^2\mathrm{F}}{dy^2}+\frac{d^2\mathrm{F}}{dx^2}=0. \quad \ldots \ldots \ldots \ldots \quad (12)$$

When the solid is symmetrical at either side of the axis of x (as it is in all the cases that will be considered in this paper), the axis of x itself, so far as it lies beyond the outline of the solid, is a water-line. Hence it is necessary that the equation of that axis, viz.

$$\left.\begin{array}{c} y=0, \\ \text{should be one of the solutions of the equation} \\ b=y+\mathrm{F}(x, y)=0, \end{array}\right\} \quad \ldots\ldots\ldots \quad (13)$$

and consequently that F should vanish with y.

The vanishing of F when $y=\infty$, indicates that every straight line given by the equation $y=b$ either forms part of, or is an asymptote to, a water-line curve.

The vanishing of F when $x=\infty$, indicates that the further the water-lines are from the generating solid, the more nearly they approximate to parallel straight lines.

Every water-line curve is itself the outline of a solid capable of moving smoothly through a liquid.

(5.) *Water-Line Curves generated by a Circle, or Cyclogenous Neoïds.*—Conceive that a circular cylinder of indefinite height, and of the radius l, described about the axis of z, moves through the liquid along the axis of x. Then it is already known that the general equation of the water-line curves is the following,

$$b=y\left(1-\frac{l^2}{x^2+y^2}\right), \quad \ldots\ldots\ldots \quad (14)$$

giving a series of curves of the third order. When $b=0$ this equation resolves itself into two, viz.

$$y=0; \quad x^2+y^2=l^2;$$

the first of which represents the axis of x, and the second the circular outline of the cylinder. For each other value of b, equation (14) represents a curve having two branches: one of them is an oval, contained within the circle, and not relevant to the problem in question; the other, being the real water-line, is convex in the middle and concave towards the ends, and has for an asymptote in both directions the straight line $y=b$.

For brevity's sake, let $x^2+y^2=r^2$. Then the component velocities of a particle of water *relatively to the solid* are given by the equations

$$\left.\begin{array}{l} \frac{u}{c}=\frac{db}{dy}=1-\frac{l^2}{r^2}+\frac{2l^2y^2}{r^4}=1+\frac{l^2(y^2-x^2)}{r^4}, \\ \frac{v}{c}=-\frac{db}{dx}=-\frac{2l^2xy}{r^4}, \end{array}\right\} \quad \ldots\ldots\ldots \quad (15)$$

and the square of their resultant by the equation

$$\frac{u^2+v^2}{c^2}=\left(1-\frac{l^2}{r^2}\right)^2+\frac{4l^2y^2}{r^4}; \quad \ldots\ldots\ldots \quad (16)$$

while the component and resultant velocities *relatively to still water* are given by the

following equations:—

$$\frac{u}{c}-1=\frac{l^2(y^2-x^2)}{r^4};\quad \frac{v}{c}=-\frac{2l^2xy}{r^4};\quad \frac{\sqrt{\{(u-c)^2+v^2\}}}{c}=\frac{l^2}{r^2}.\quad . \quad . \quad . \quad . \quad . \quad (17)$$

As a convenient name for water-line curves of this sort, it is proposed to call them *Cyclogenous Neoïds*, that is, *ship-shape curves generated from a circle.*

The water-line surfaces generated by a sphere are known; but no use will be made of them in this paper. (See paper by Dr. HOPPE, Quart. Journ. Math., March 1856.)

SECTION II.—*Properties of Water-line Curves generated from Ovals, or Oögenous Neoïds.*

(6.) *Derivation of other Water-Line Curves from Cyclogenous Neoïds.*—When a form of the function F has been found which satisfies equation (12) of art. 4 (that is to say, which fulfils the condition of liquidity), an endless variety of other forms of that function possessing the same property may be derived from the original form by differentiation and integration.

The original form, and also the derived forms, must possess the properties of vanishing for $x=\infty$ and for $y=\infty$, and of becoming $=0$ or a constant for $y=0$. The first of those properties excludes trigonometrical functions, and consequently exponential functions also, which are always accompanied by trigonometrical functions, and leaves available functions of the nature of potentials. The second property excludes derivation by means of differentiation and integration with respect to y, and leaves available differentiation and integration with respect to x.

The original form of the function F which will be used in this paper is that appropriate to cyclogenous neoïds, or water-line curves generated from a circle, as given in equation (14) of art. 5, viz.—

$$F=\frac{y}{r^2}\times \text{ constant.}$$

When one or more differentiations with respect to x are performed on this function, and the results substituted for F in equation (10), there are obtained curves which are real water-lines, but which are not suitable for the figures of ships, some of them being lemniscates, others shaped like an hour-glass, and others looped and foliated in various ways. It is otherwise as regards integration with respect to x; for that operation, being performed once, gives the expression for the ordinate in a class of curves all of which resemble possible forms of ships, and which are so various in their proportions, that every form of ships' water-lines which has been found to succeed in practice may be closely imitated by means of them. As that class of curves consists of certain ovals, and of other water-lines generated from those ovals, it is proposed to call them *Oögenous Neoïds* (from Ὠογενής).

(7.) *General Equation of Oögenous Neoïds.*—The integration with respect to x, already referred to, is performed as follows:—The coordinates of a particle of water being x and y, let x' denote the position of a moveable point in the axis of x: then the function to be integrated is

$$\frac{y}{(x-x')^2+y^2}$$

for all values of x' between two arbitrary limits. Let $2a$ denote the distance between those limits: the most convenient position for the origin of coordinates is midway between them, so as to make the limits

$$x'=+a, \quad x'=-a \text{ respectively.}$$

Then the following is the integral sought:

$$\int_{x'=-a}^{x'=+a} \frac{y\,dx'}{(x-x')^2+y^2} = \tan^{-1}\frac{a-x}{y} + \tan^{-1}\frac{a+x}{y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This quantity evidently denotes *the angle contained between two lines drawn from the point* (x, y) *to the points* $(+a, 0)$ *and* $(-a, 0)$. For brevity's sake, in the sequel that angle will be occasionally denoted by θ; the points $(+a, 0)$ and $(-a, 0)$ will be called the *foci*; and their distance a from the centre will be called the *excentricity*.

Substituting this integral in the general equation (10), we find, for the water-line curves now under consideration, the following equation, which is the general equation of *oögenous neoïds*:—

$$b=y-f\theta=y-f\left(\tan^{-1}\frac{a-x}{y} + \tan^{-1}\frac{a+x}{y}\right). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The coefficient f denotes an arbitrary length, which will be called the *parameter*.

(8.) *Geometrical Meaning of that Equation.*—The equation (19) represents a curve at each point of which the excess $(y-b)$ of the ordinate (y) above a certain minimum value (b) is proportional to the angle (θ) contained at that point between two straight lines drawn to the two foci. Except when $b=0$, the curve has an asymptote at the distance b from the axis of x, and parallel to that axis. Since the value of b is not altered by reversing the signs of x, and is only changed from positive to negative by reversing the sign of y, it follows that each curve consists of two halves, symmetrical about the axis of y; and that there are pairs of curves, symmetrical about the axis of x.

In Plate VIII. fig. 1, therefore, which represents a series of such curves, one quadrant only of the space round the origin or centre O is shown, the other three quadrants being symmetrical. A is one of the foci, at the distance $OA=a$ from the centre; the other focus, not shown in the figure, is at an equal distance from the centre in the opposite direction. BL is one quadrant of the primitive oval; and the wave-like curves outside of it are a series of water-lines generated from it, having for their respective asymptotes the series of straight lines parallel to OX, and whose distances from OX are a series of values of b.

The equation (19) embraces also a set of curves contained within the oval, and all traversing the two foci; but as these curves are not suited for the forms of ships' water-lines, no detailed description of them needs be given.

(9.) *Properties of Primitive Oval Neoïds.*—When in equation (19) b is made $=0$, so that the equation becomes

$$y-f\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

there are two solutions; one of which, viz. $y=0$, represents the axis of x, agreeably to

the condition stated in article 4, equations (13). The other solution represents the oval LB.

The greater semiaxis of that oval, OL, will be called the *base* of the series of water-lines generated by the oval, and denoted by l; its value is found as follows:

$$\frac{db}{dy}=1+f\frac{d}{dy}\left(\tan^{-1}\frac{y}{a-x}+\tan^{-1}\frac{y}{a+x}\right)=1+f\left\{\frac{a-x}{(a-x)^2+y^2}+\frac{a+x}{(a+x)^2+y^2}\right\};$$

but at the point L we have

$$x=l;\ y=0;\ \frac{db}{dy}=0;$$

and therefore

$$0=1+f\left(\frac{1}{a-l}+\frac{1}{a+l}\right);$$

whence

$$l^2=a^2+2af. \quad \ldots \quad (21)$$

To find the *parameter* f when the base l and excentricity a are given, we have the formula

$$f=\frac{l^2-a^2}{2a}. \quad \ldots \quad (22)$$

The half-breadth, or minor semiaxis of the oval, $OB=y_0$, is the root of the following transcendental equation, found by making $x=0$ in equation (19),

$$y_0-2f\tan^{-1}\frac{a}{y_0}=0, \quad \ldots \quad (23)$$

which may be otherwise written as follows:—

$$\tan\frac{y_0}{2f}-\frac{a}{y_0}=0. \quad \ldots \quad (23\text{ A})$$

When the minor semiaxis y_0 and excentricity a are given, the parameter f is found by the equation

$$f=\frac{y_0}{2\tan^{-1}\frac{a}{y_0}}; \quad \ldots \quad (24)$$

and thence the base l can be computed by equation (21).

When the base l and half-breadth y_0 are given, the excentricity a is found by solving the following transcendental equation:—

$$ay_0-(l^2-a^2)\tan^{-1}\frac{a}{y_0}=0. \quad \ldots \quad (24\text{ A})$$

An oval neoïd differs from an ellipse in being fuller towards the ends and flatter at the sides; and that difference is greater the more elongated the oval is.

(10.) *Varieties of Oval Neoïds, and extreme cases.*—The excentricity a may have any value, from nothing to infinity; and the base l may bear to the half-breadth y_0 any proportion, from equality to infinity. When the excentricity $a=0$, the two foci coalesce with the centre O; the base l becomes equal to the half-breadth b; the oval becomes a

circle of the radius l; and the water-lines generated by it become cyclogenous neoïds, already described in article 5.

As the excentricity increases, the oval becomes more elongated. In Plate IX. fig. 3, PL is an oval whose length is to its breadth as $\sqrt{3} : 1$, its focus being at A_0. The oval BL in Plate VIII. fig. 1 is more elongated, its length being to its breadth as 17 : 6 nearly. When the excentricity is infinite, the centre O and the further focus go off to infinity, leaving only one focus. The parameter f becomes equal to the focal distance LA. The oval is converted into a curve bearing the same sort of analogy to a parabola that an oval neoïd bears to an ellipse*; but instead of spreading to an infinite breadth like a parabola, it has a pair of asymptotes parallel to the axis of x, and at the distance $\pm\pi f$ to either side of it; and each generated water-line has two parallel asymptotes, at the respective distances b and $b+\pi f$ from the axis of x. The properties of these curves may be easily investigated by placing the origin of coordinates at the focus A, and substituting, in equation (19), $\tan^{-1}\frac{y}{x}$ for θ; but as their figure is not suitable for ships' water-lines, it is unnecessary here to discuss them in detail; and the same may be said of a class of curves analogous to hyperbolas, whose equation is formed by putting $-$ instead of $+$ between the two terms of the right-hand member of equation (18).

(11.) *Graphic Construction of Oval and Oögenous Neoïds.*—For the sake of distinctness, the processes of drawing these curves are represented in two figures,—fig. 2 showing the preliminary, and fig. 1 the final processes (see Plate VIII.).

The axis OY is to be divided into equal parts of any convenient length (which will be denoted by δy in what follows), and through the divisions are to be drawn a series of straight lines parallel to OX. (It is convenient to print those lines from a copper-plate divided and ruled by machinery.) They are shown in fig. 1 only, and not in fig. 2, to avoid confusion.

Suppose, now, that the problem is as follows:—*The base* OL *and excentricity* OA *being given, it is required to construct the oval neoïd and the water-lines generated by it.*

Through the focus A (Plate VIII. fig. 2) draw AD perpendicular to OX; about O, with the radius OL, describe the circular arc LD, cutting AD in D; from D draw DE perpendicular to OD, cutting OX in E; then (as equation (22) shows) AE will be $=2f$, the *double parameter*.

About A, with the radius $AE=2f$ thus found, describe a circle cutting AD in F. Then commencing at F, lay off on that circle a series of arcs, each equal to $2\delta y$ (the double of the length of the equal divisions of the axis OY). Through the points of division of the circle draw a series of radii, AG_1, AG_2, &c., cutting the axis OY in a series of points (some of which, from G_3 to G_{12}, are marked in fig. 2)†. (These radii make, with the line AD, a series of angles, $\frac{\delta y}{f}$, $\frac{2\delta y}{f}$, $\frac{3\delta y}{f}$, &c.)

* This curve is identical with the quadratrix of TSCHIRNHAUSEN.

† When the parameter is small, it is sometimes advisable to use a circle (such as a protractor) with a radius

Then about each of the points in the axis OY thus found, with the outer leg of the compasses starting from the focus A, describe a series of circles (shown in Plate VIII. fig. 1) AC_1, AC_2, AC_3, &c.

Each of those circles traverses the two foci; and the equation of any one of them is

$$-F=f\theta=n\delta y, \quad \ldots \ldots \ldots \ldots \quad (25)$$

where θ denotes the angle made at any point of the circle by straight lines drawn to the two foci, and n has the series of values 1, 2, 3, &c. Since F, as explained in article 4, is the characteristic function for the motion of the liquid particles relatively to still water, it is plain that each of the circles for which F= constant is a tangent to the directions of motion of all the particles that it traverses.

The paper is now covered, as in fig. 1, with a network made by a series of straight lines whose equations are of the form $y=n'\delta y$, crossed by a series of circles whose equations are of the form $f\theta=n\delta y$.

Consequently any curve drawn like those in Plate VIII. fig. 1, diagonally through the corners of the quadrangles of that network, will have for its equation

$$y-f\theta=(n'-n)\delta y=b,$$

and will accordingly be an oögenous neoïd, having for its asymptote the line $y=b$.

The primitive oval is drawn by starting from the point L, and traversing the network diagonally. As many curves as are required can be drawn by the eye with great precision, and the whole process is very rapid and easy (see Appendix).

When the problem is *with a given base and excentricity to draw an oögenous neoïd through a given point in the axis* OY, such as P, the process is modified as follows:—The axis OY must be so divided that P shall be at a point of division. Then, up to the describing of the circle about A with the radius AE, the process is the same as before. Then join AP (Plate VIII. fig. 2), and draw Ag making the angle PAg=APO, and cutting the axis OY in a point (such as G_{10}) which will be the centre of the circle traversing A and P. Then on the circumference of the circle about A, from g towards F, lay off a series of arcs each $=2\delta y$; through the points of division draw radii cutting the axis OY in the points G_9, G_8, &c., and complete the process as before.

(12.) *Graphic Construction of Cyclogenous and Parabologenous Neoïds.*—When the excentricity vanishes and the oval becomes a circle, all the circles composing the network become tangents to OX at the point O. They pass through the points where the primitive circular water-line is cut by the equidistant parallel lines. Their radii are in harmonic progression; the equation of any one of them is of the form

$$-F=\frac{l^2y}{x^2+y^2}=n\delta y, \quad \ldots \ldots \ldots \ldots \quad (26)$$

which is a larger multiple of the parameter than double, the length of the divisions being increased in the same proportion; or the points on the axis OY may be laid down by means of their distances from O, calculated by the formula $OG=a \cdot \cot\theta$.

n having the series of values 1, 2, 3, &c.; and its radius is given by the formula

$$\frac{l^2}{n\delta y}. \quad \ldots \ldots \ldots \quad (26\ \text{A})$$

When there is but one focus, as in the infinitely long curve described in article 10, the network of circles is changed into a set of straight lines radiating from the focus, and making with AX the series of angles given by the formula

$$f\theta = n\delta y. \quad \ldots \ldots \ldots \quad (27)$$

(13.) *Component and Resultant Velocities of Gliding.*—The component and resultant velocities with which the liquid particles glide along the water-lines are given by the following equations, in terms of the excentricity a, the parameter f, and the coordinates:—

$$\left.\begin{aligned} &\frac{u}{c} = \frac{db}{dy} = 1 + \frac{f(a-x)}{(a-x)^2+y^2} + \frac{f(a+x)}{(a+x)^2+y^2}; \quad \frac{v}{c} = -\frac{db}{dx} = -\frac{fy}{(a-x)^2+y^2} + \frac{fy}{(a+x)^2+y^2}; \\ &\frac{u^2+v^2}{c^2} = \frac{db^2}{dy^2} + \frac{db^2}{dx^2} = 1 + \frac{2f(a-x)}{(a-x)^2+y^2} + \frac{2f(a+x)}{(a+x)^2+y^2} + \frac{4f^2a^2}{\{(a-x)^2+y^2\}.\{(a+x)^2+y^2\}}. \end{aligned}\right\} \quad (28)$$

At the point of greatest (breadth that is, at the axis of y) these expressions take the following values:—

$$\frac{u_0}{c} = 1 + \frac{2fa}{a^2+y_0^2} = 1 + \frac{l^2-a^2}{a^2+y_0^2} = \frac{l^2+y_0^2}{a^2+y_0^2}; \quad \frac{v_0}{c} = 0. \quad \ldots \ldots \quad (28\ \text{A})$$

These equations are applicable to a *whole series of water-lines* (such as those shown in fig. 1), including the generating oval, and are the best suited for solving questions relating to such a series.

But when *one particular water-line* is in question, it is sometimes more convenient to use another set of equations, formed from the equations (28) by the aid of the following substitutions, in which θ, as before, denotes $\frac{y-b}{f}$:—

$$\left.\begin{aligned} &\{(a-x)^2+y^2\}.\{(a+x)^2+y^2\} = \frac{4a^2y^2}{\sin^2\theta}; \\ &\{(a-x)^2+y^2\} + \{(a+x)^2+y^2\} = 2a^2+2x^2+2y^2 = 4a^2+4ay\cot\theta; \\ \therefore\quad &x^2+y^2 = a^2+2ay\cot\theta; \quad x = \sqrt{\{a^2-y^2+2ay\cot\theta\}}. \end{aligned}\right\} \quad (29)$$

These substitutions being made in the equations (28), give the following results:—

$$\left.\begin{aligned} \frac{u}{c} &= 1 + \frac{f}{a}\sin^2\theta - \frac{f}{y}\cdot\cos\theta\sin\theta = 1 + \frac{f}{2a} - \frac{f\cos 2\theta}{2a} - \frac{f\sin 2\theta}{2y}; \\ \frac{v}{c} &= -\frac{fx}{ay}\sin^2\theta = -\frac{f}{ay}\sqrt{\{a^2-y^2+2ay\cot\theta\}}\sin^2\theta; \\ \frac{u^2+v^2}{c^2} &= 1 + \frac{2f}{a}\sin^2\theta - \frac{2f}{y}\cos\theta\sin\theta + \frac{f^2}{y^2}\sin^2\theta \\ &= 1 + \frac{f}{a} + \frac{f^2}{2y^2} - \left(\frac{f}{a} + \frac{f^2}{2y^2}\right)\cos 2\theta - \frac{f}{y}\sin 2\theta. \end{aligned}\right\} \quad \ldots \quad (30)$$

(14.) *Trajectories of Normal Displacement, and of Swiftest and Slowest Gliding.*—By the "trajectory of normal displacement" is meant a curve traversing all the points in a series of water-lines at which the directions of motion of the liquid particles relatively to still water are perpendicular to the water-lines; or, speaking geometrically, a curve traversing all the points at which the circles AC_1, AC_2, &c. of fig. 1, Plate VIII. cut the water-lines at right angles. To find the form of that trajectory, it is sufficient to make

$$\frac{u^2+v^2}{c^2}-\frac{u}{c}=0, \quad \ldots \ldots \ldots \ldots \quad (31)$$

employing the values of these ratios given by the equations (28). This having been done, it appears, after some simple reductions, that the equation of the *trajectory of normal displacement* is the following,

$$x^2-y^2=l^2, \quad \ldots \ldots \ldots \ldots \quad (32)$$

being that of a rectangular hyperbola LM, fig. 1, having its vertex at L, and its centre at O. Hence that curve is *similar for all oögenous and cyclogenous neoïds whatsoever*, being independent of the excentricity, and is identical for all oögenous and cyclogenous neoïds having the same base l.

By the "trajectory of swiftest and slowest gliding" is meant a curve traversing every point in a series of water-lines at which the velocity of gliding, $\sqrt{u^2+v^2}$, is a maximum or a minimum for the water-line on which that point is situated. To find the equation of that curve, it is necessary to solve the following equation,

$$\frac{d}{cdt}\left(\frac{u^2+v^2}{c^2}\right)=\left(\frac{u}{c}\cdot\frac{d}{dx}+\frac{v}{c}\cdot\frac{d}{dy}\right)\left(\frac{u^2+v^2}{c^2}\right)=0, \quad \ldots \ldots \quad (33)$$

the expression employed for $\frac{u^2+v^2}{c^2}$ being that given by the third of the equations (28).

After a tedious but not difficult process of differentiation and reduction, which it is unnecessary to give in detail, an equation is found which resolves itself into three factors, viz.

$$x=0, \quad \ldots \ldots \ldots \ldots \ldots \quad (34)$$

being the equation of the axis OY, and

$$\sqrt{x^2+y^2}+y\pm\sqrt{l^2+y^2}=0, \quad \ldots \ldots \ldots \quad (35)$$

being the equations of the two branches LN and LP of a curve of the fourth order. This curve, too, is independent of the excentricity, and therefore *similar for all oögenous and cyclogenous neoïds whatsoever*, and identical for those having the same base l. It has also the following properties:—The straight line joining L with P makes an angle of 30° with the axis OX; there are a pair of straight asymptotes through O, making angles of 30° to either side of OX; and the two branches of the curve cut OX in the point L, at angles of 45°.

(15.) *Graphic Construction of those Trajectories.*—The curves described in the preceding article are easily and quickly constructed, with the aid of the series of equidistant lines parallel to OX, as follows:—In fig. 2, Plate VIII., let ST be any one of

those lines. With the distance SL in the compasses, lay off SH on that line; H will be a point in the hyperbola LM. Also from S lay off, on the axis of y, SI and SJ, each equal to the same distance SL. About the centre O, with the radius OI, draw a circular arc cutting ST in K; this will be a point in the branch LN. About the centre O, with the radius OJ, draw a circular arc cutting ST in k; this will be a point in the branch LP.

(16.) *Properties of the Trajectory of Swiftest and Slowest Gliding.*—The branch LN traverses a series of points of slowest gliding, where the water-lines are furthest apart; the branch LP traverses a set of points of swiftest gliding, where the water-lines are closest together; from O to P the axis of y traverses points of slowest gliding, and beyond P, points of swiftest gliding.

Hence every complete oögenous neoïd which cuts the axis of y between O and P, contains two points of swiftest and three of slowest gliding; and every complete oögenous or cyclogenous neoïd which cuts the axis of y at or beyond P, contains only one point of swiftest and two of slowest gliding.

(17.) *Water-Lines of Smoothest Gliding, or Lissoneoïds.*—At the point P itself, situated at the distance

$$\mathrm{OP}=\frac{l}{\sqrt{3}} \quad \ldots \ldots \ldots \ldots \quad (36)$$

from the centre, two maxima and a minimum of the velocity of gliding coalesce; and therefore not only the first, but the second and third differential coefficients of the velocity of gliding vanish; from which it follows that the velocity of gliding changes more gradually on those water-lines which pass through the point P, than on any other class of oögenous or cyclogenous neoïds.

It is proposed therefore to call this class of water-lines *Lissoneoïds* (from λισσός).

The oval neoïd whose length is to its breadth as $\sqrt{3}:1$ is itself a lissoneoïd; and every series of water-lines generated by an oval *more elongated* than this contains one lissoneoïd; for example, in the series of water-lines shown in fig. 1, the lissoneoïd is marked PQ.

The excentricity of the oval lissoneoïd is computed by solving equation (24 A) of article 9, when $y_0=\frac{l}{\sqrt{3}}$; and it is found to be

$$a=\cdot 732 l, \text{ or nearly } (\sqrt{3}-1)l. \quad \ldots \ldots \quad (36\text{ A})$$

By giving the excentricity values ranging from $\cdot 732l$ to l, there are produced a series of lissoneoïds ranging from the oval PL in fig. 3, Plate IX., whose focus is at A_0, to the straight line PN, whose focus coalesces with L. PQ_1, PQ_2, and PQ_3 are specimens of the intermediate forms, having their foci respectively at A_1, A_2, and A_3. For a reason which will be explained in Section III., those curves are not shown beyond the trajectory of slowest gliding.

The *greatest speed of gliding*, for a lissoneoïd, is found by making $y_0^2=\frac{l^2}{3}$ in equation (28 A) of article 13; that is to say,

$$\frac{u_0}{c}=\frac{4l^2}{3a^2+l^2}. \quad \ldots \ldots \ldots \ldots \quad (37)$$

(18.) *Orbits of the Particles of Water.*—The general expressions for the components of the velocity of a liquid particle relatively to still water have been given in equation (11) of article 4; and to apply those to the case of oögenous neoïds, it is only necessary to modify the equations (28) of article 13, by introducing the expression for $\frac{u-c}{c}$ instead of that for $\frac{u}{c}$, as follows:—

$$\left.\begin{array}{c} \dfrac{u-c}{c}=\dfrac{(l^2-a^2)\,.\,(a^2-x^2+y^2)}{\{(a-x)^2+y^2\}\,.\,\{(a+x)^2+y^2\}}\,; \quad \dfrac{v}{c}=\dfrac{-2(l^2-a^2)xy}{\{(a-x)^2+y^2\}\,.\,\{(a+x)^2+y^2\}}\,; \\ \dfrac{(u-c)^2+v^2}{c}=\dfrac{(l^2-a^2)^2}{\{(a-x)^2+y^2\}\,.\,\{(a+x)^2+y^2\}}\,. \end{array}\right\} \quad . \quad . \quad (38)$$

From the last of these equations it appears that *the velocity of a particle relatively to still water is inversely as the product of its distances from the two foci.*

The only other investigation which will here be made respecting the orbit of a particle of water, is that of the relation between its direction and curvature at a given point, and its ordinate y.

It has already been explained, in article 11, that the direction of motion of a particle is a tangent to a circle traversing it and the two foci. The radius of that circle is

$$\frac{a}{\sin\frac{y-b}{f}}=\frac{a}{\sin\theta}\,;$$

and if φ be taken to denote the angle which the direction of the particle's motion relatively to still water makes with the axis of x, it is easily seen that

$$\cos\varphi=\cos\theta-\frac{y}{a}\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

While that angle undergoes the increment $d\varphi$, the particle moves through an arc of its orbit whose length is $\frac{dy}{\sin\varphi}$; consequently the curvature of that orbit at the arc in question is

$$\frac{1}{\varrho}=\frac{\sin\varphi d\varphi}{dy}=-\frac{d\,.\cos\varphi}{dy}=\left(\frac{1}{f}+\frac{1}{a}\right)\sin\theta+\frac{y}{fa}\cos\theta=\frac{2}{l^2-a^2}\cdot\left\{\frac{l^2+a^2}{2a}\cdot\sin\theta+y\cos\theta\right\}. \quad (40)$$

For cyclogenous neoïds, we obtain the value of this expression by making

$$\sin\theta=\frac{y-b}{f},\ \cos\theta=1,$$

substituting l^2-a^2 for $2fa$, and then making $a=0$; the result being as follows,

$$\frac{1}{\varrho}=\frac{4}{l^2}\left(y-\frac{b}{2}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40\text{ A})$$

that is to say, the *curvature of the orbit varies as the distance of the particle from a line parallel to the axis of* x, *and midway between that axis and the undisturbed position of the particle.* This is the property of the looped or coiled elastic curve; therefore, *when*

the water-lines are cyclogenous, the orbit of each particle of water forms one loop of an elastic curve.

The general appearance of such an orbit is shown in fig. 6, Plate VIII. The arrow D shows the direction of motion of the solid body. The dotted line AC is supposed to be at the distance b from the axis of x. The particle starts from A, is at first pushed forwards, then deviates outwards and turns backwards, moving directly against the motion of the solid body as it passes the point of greatest breadth, as shown at B. The particle then turns inwards, and ends by following the body, and coming to rest at C, in advance of its original position.

When the water-lines are oögenous, the equations (39) and (40) show that the orbit is of the same general character with the looped elastic curve in fig. 6, but differs from it in detail to an extent which is greater the greater the excentricity a; and the difference consists mainly in a flattening of the loop, so as to make it less sharply curved at B.

When the excentricity increases without limit, the orbit approximates indefinitely to a "curve of pursuit," for which

$$\varphi=\theta,\ \frac{1}{\varrho}=\frac{\sin\theta}{f}. \qquad (40\text{B})$$

(19.) *Trajectory of Transverse Displacement.—Of Speed of Gliding equal to Speed of Ship.—Orthogonal Trajectories.*—The trajectories described in this article differ from those described in articles 14, 15, and 16 by being dependent upon the excentricity, and therefore not similar for all sets of oögenous neoïds.

By the "trajectory of transverse displacement" is meant the curve traversing all the points at which the liquid particles are moving at right angles to the axis OX, relatively to still water. It is determined from the first of the equations (28), by making

$$\frac{u}{c}-1=0;$$

from which is easily deduced the following equation,

$$x^2-y^2=a^2, \qquad (41)$$

being that of a rectangular hyperbola, with its centre at O and its vertex at the focus A.

The trajectory of the points where the speed of gliding is equal to the speed of the solid body, is found from the third of the equations (28), by making

$$\frac{u^2+v^2}{c^2}-1=0.$$

Its equation is

$$x^2-y^2=\frac{l^2+a^2}{2}, \qquad (42)$$

being that of a rectangular hyperbola, with its centre at O and its vertex between A and L, at a distance from O equal to half the hypothenuse of a right-angled triangle whose other sides are equal to the base and the excentricity respectively.

Let $q=$ constant be the equation of one out of an indefinite number of orthogonal

trajectories to a set of oögenous neoïds. The function q, as is well known, must satisfy the equation

$$\frac{dq}{dx}\cdot\frac{db}{dx}+\frac{dq}{dy}\cdot\frac{db}{dy}=0.$$

Referring to equation (19) of article 7 for the value of b, it is easily seen that this condition is fulfilled by the following function,

$$q=x+\frac{f}{2}\text{ hyp. log. }\frac{(a+x)^2+y^2}{(a-x)^2+y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

which has also the following properties,

$$\frac{dq}{dx}=\frac{db}{dy}=\frac{u}{c};\ \frac{dq}{dy}=-\frac{db}{dx}=\frac{v}{c};\ \frac{d^2q}{dx^2}+\frac{d^2q}{dy^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Every orthogonal trajectory has a straight asymptote parallel to the axis of y, and expressed by the equation $x=q$.

The perpendicular distance between two consecutive orthogonal trajectories, like that between two consecutive water-lines, is inversely proportional to the velocity of gliding; hence, if a complete set of orthogonal trajectories were drawn on fig. 1, they would divide it into a network of small rectangles, the dimensions and area of any one of which would be expressed as follows:—

$$\frac{cdb}{\sqrt{u^2+v^2}}\times\frac{cdq}{\sqrt{u^2+v^2}}=\frac{c^2dbdq}{u^2+v^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

For a series of cyclogenous neoïds, the equation of the orthogonal trajectories takes the following form,

$$q=x\left(1+\frac{l^2}{x^2+y^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45\text{A})$$

(20.) *Disturbances of Pressure and Level.*—Let h denote the *head* at a given particle of liquid, being the sum of its elevation above a fixed level and of its pressure, expressed in units of height of the liquid itself. In a mass of liquid which is at rest, the head has a uniform value for every particle of the mass; let that value be denoted by h_0. Then when the mass of liquid is in the state of motion produced by the passage of a solid through it, the head at each particle, according to well-known principles, undergoes the change expressed by the following equation,

$$h-h_0=\frac{c^2-u^2-v^2}{2g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

being the height due to the difference between the squares of the speed of the solid body and of the speed of gliding; and in an open mass of water with a vessel floating in it, that change will take place by alterations in the level of surfaces of equal pressure. The trajectory of slowest gliding, LN (Plate VIII. fig. 1), will mark the summit of a swell thus produced, and so also will the axis of y between O and P; while the trajectory of swiftest gliding OP, and the axis of y beyond P, will mark the bottom of a hollow. These are the principal vertical disturbances, which, throughout this investigation, have been assumed

to be so small, compared with the dimensions of the body, as not to produce any appreciable error in the consequences of the supposition of motion in plane layers.

(21.) *Integral on which the Friction depends.*—Suppose a portion of an oögenous neoïd to be taken for the water-line of part of the side of a vessel, which part is of the depth δz, and that the resistance arising from friction between the water and the vessel is to be expressed—the law of that friction being, that it varies as the square of the velocity of gliding, and as the extent of rubbing surface.

That resistance is to be found (as already explained in a paper on Waves, published in the Philosophical Transactions for 1863) by determining the *work performed* in a second in overcoming friction, and dividing by the speed of the vessel; for thus is taken into account not only the direct resistance caused by the longitudinal component of the friction, but the resistance caused indirectly through the increase of pressure at the bow, and diminution of pressure at the stern, assuming the vertical disturbance to be unimportant.

Then for a part of the water-line which measures longitudinally dx, the extent of surface is

$$\delta z \cdot \frac{\sqrt{u^2+v^2}}{u} dx;$$

the friction on the unit of surface is

$$\frac{\mathrm{KW}(u^2+v^2)}{2g},$$

where W is the weight of a unit of volume of water, and K a coefficient of friction; and that friction has to be overcome through the distance $\sqrt{u^2+v^2}$, while the vessel advances through the distance c, giving as a factor

$$\frac{\sqrt{u^2+v^2}}{c}.$$

Those three factors being multiplied together, and the result put under the sign of integration, give the following expression for the resistance,

$$\mathrm{R}=\frac{\mathrm{KW}c^2}{2g}\delta z \cdot \int \left(\frac{u^2+v^2}{c^2}\right)^2 \cdot \frac{c}{u} dx. \quad \text{(46 A)}$$

Another form of expression for the same integral is obtained by putting $\frac{c}{v} dy$ or $f\frac{c}{v} d\theta$ instead of $\frac{c}{u} dx$; and a third form by putting for the elementary area of the rubbing surface the following value,

$$dz \cdot \frac{c}{\sqrt{u^2+v^2}} dq;$$

where dq is the distance between the asymptotes of a pair of orthogonal trajectories, as explained in article 19. This gives for the resistance

$$\mathrm{R}=\frac{\mathrm{KW}c^2}{2g}\delta z \cdot \int \frac{u^2+v^2}{c^2} \cdot dq. \quad \text{(47)}$$

In preparing these formulæ for integration, it is necessary to express the function to be integrated in terms of constants and of the independent variable only, x, y, θ, or q, as the case may be; for example, if y or θ is the independent variable, the expression of the function to be integrated is to be taken from the equations (30) of article 13.

Owing to the great complexity of that function, its exact integration presents difficulties which have not yet been overcome, although a probable approximate formula for the resistance has been arrived at by methods partly theoretical and partly empirical, as to which some further remarks will be made in the third section of this paper*.

There is one particular case only in which the exact integration of equation (46 A) is easy, that of a complete circular water-line of the radius l; and the result is as follows:—

$$\mathrm{R}=\frac{\mathrm{KW}c^2}{2g}\delta z\times 21\tfrac{1}{3}l. \qquad (48)$$

(22.) *Statement of the General Problem of the Water-Line of least Friction.*—It is evident that, by introducing under the sign of integration in equation (18) of article 7 an arbitrary function of x', the integral may be made capable of representing an arbitrary function of x and y, and will still satisfy the condition of perfect liquidity; and thus the equation

$$b=y+\int_{-a}^{+a}\frac{y\varphi(x')dx'}{(x-x')^2+y^2}=0 \qquad (48\text{ A})$$

may be made to represent an arbitrary form of primitive water-line.

To find therefore, by the calculus of variations, the water-line enclosing a given area which shall have the least friction, will require the solution of the following problem:— To determine the function $\varphi(x')$ so that, with a fixed value of the integral $\int xdy$, the integral in equation (46 A) shall be a minimum.

(22 A.) *Another Class of Plane Water-Line Equations.*—A mode of expressing the conditions of the flow of water in plane layers past a solid, differing in form from that made use of in the preceding parts of this paper, consists in taking for independent variables, not the coordinates of the water-lines themselves, x and y, but the coordinates of their asymptotes (b), and of the asymptotes of their orthogonal trajectories (q). These new variables are connected with x and y, and with the velocity of gliding, by the following equations:—

$$\frac{u^2+v^2}{c^2}=\frac{dq}{dx}\cdot\frac{db}{dy}-\frac{dq}{dy}\cdot\frac{db}{dx}=\frac{1}{\frac{dx}{dq}\cdot\frac{dy}{db}-\frac{dy}{dq}\cdot\frac{dx}{db}}. \qquad (49)$$

It can be shown that in order to satisfy the condition of liquidity we must have

$$x=\frac{d\psi}{db},\quad y=\frac{d\psi}{dq}; \qquad (50)$$

* See the Civil Engineer and Architect's Journal for October 1861, the Philosophical Transactions for 1863, the Transactions of the Institution of Naval Architects for 1864, and a Treatise on Shipbuilding, published in 1864.

where ψ denotes a function of b and q, such that

$$\frac{d^2\psi}{db^2}+\frac{d^2\psi}{dq^2}=0; \quad \ldots \ldots \ldots \ldots \quad (51)$$

and consequently that

$$\frac{c^2}{u^2+v^2}=\left(\frac{d^2\psi}{dbdq}\right)^2+\left(\frac{d^2\psi}{db^2}\right)^2. \quad \ldots \ldots \ldots \quad (52)$$

The curves to which this method of investigation leads are inferior to oögenous neoïds as water-lines for ships, because they have comparatively sharp curvature amidships, which causes them to have small capacity for their length and breadth, and would give rise to comparatively sudden changes in the speed of gliding. They will therefore not be further discussed in the present paper, except to state that the simplest of them is the well-known cissoid.

SECTION III.—*Remarks on the Practical Use of Oögenous Water-Lines.*

(23.) *Previous Systems of Water-Lines.*—Owing principally to the great antiquity of the art of shipbuilding, and the immense number of practical experiments of which it has been the subject, that part of it which relates to the forms of water-lines has in many cases attained a high degree of excellence through purely empirical means. Excellence attained in that manner is of an uncertain and unstable kind; for as it does not spring from a knowledge of general principles, it can be perpetuated by mere imitation only.

The existing forms of water-lines, whose merits are known through their practical success, constitute one of the best tests of a mathematical theory of the subject; for if that theory is a sound one, it will reproduce known good forms of water-line; and if it is a comprehensive one, it will reproduce their numerous varieties, which differ very much from each other.

The geometrical system of CHAPMAN for constructing water-lines is wholly empirical; it consists in the use of parabolas of various orders, chosen so as to approximate to figures that have been found to answer in practice, and it has no connexion with any mechanical theory of the motion of the particles of water.

The first theory of ships' water-lines which was at once practically useful, and based on mechanical principles, was that of Mr. SCOTT RUSSEEL, explained in the first and second volumes of the Transactions of the Institution of Naval Architects. It consists of two parts; the first has reference to the dimensions of water-lines intended for a given maximum speed, and prescribes a certain relation between the length of those lines and the length of a natural wave which travels with that speed; the second part relates to the form of those lines, and prescribes for imitation the figures of certain natural waves, as being lines along which water is more easily displaced than along other lines. The figures thus obtained are known to be successful in practice; but it is also well known that there are other figures which answer well in practice, differing considerably from those wave-lines; and it is desirable that the mathematical theory of the subject should

embrace those figures also. It may further be observed that the figure of the solitary wave, as investigated experimentally by Mr. SCOTT RUSSELL (Reports of the British Association, 1845), and mathematically by Mr. EARNSHAW (Camb. Trans. 1845), is that of a wave propagated in a canal of small breadth and depth as compared with the dimensions of the wave, and in which particles of water originally in a plane at right angles to the direction of motion continue to be very nearly in a plane at right angles to the direction of motion, so as to have sensibly the same longitudinal velocity. This state of things is so different from the circumstances of the motions of the particles in the open sea, that it appears desirable to investigate the subject with special reference to a mass of water of unlimited breadth and depth, as has been done in the previous sections of this paper.

(24.) *Variety of Forms of Oögenous Neoïds, and their likeness to good known Forms of Water-Line.*—The water-lines generated from ovals which have been described in the second section of this paper, are remarkable for the great varieties of form and proportions which they present, and for the resemblance of their figures to those of the water-lines of the different varieties of existing vessels. There is an endless series of ovals, having all proportions of length to breadth, from equality to infinity; and each of those ovals generates an endless series of water-lines, with all degrees of fulness or fineness, from the absolute bluffness of the oval itself to the sharpness of the knife-edge. Further variations may be made by taking a greater or a less length of the curve chosen.

The ovals are figures suitable for vessels of low speed, it being only necessary, in order to make them good water-lines, that the vertical disturbance (as explained in art. 20) should be small compared with the vessel's draught of water. At higher speeds the sharper water-lines, more distant from the oval, become necessary. The water-lines generated by a circle, or "cyclogenous neoïds," are the "leanest" for a given proportion of length to breadth; and as the excentricity increases, the lines become "fuller." The lines generated from a very much elongated oval approximate to a straight middle body with more or less sharp ends. In short, there is no form of water-line that has been found to answer in practice which cannot be imitated by means of oögenous neoïds.

(25.) *Discontinuity at the Bow and Stern.—Best limits of Water-Lines.*—Amongst the endless variety of forms presented by oögenous water-lines, it may be well to consider whether there are any which there are reasons for preferring to the others. One of the questions which thus arise is the following:—Inasmuch as all the water-line curves of a series, except the primitive oval, are infinitely long and have asymptotes, there must necessarily be an abrupt change of motion at either end of the limited portion of a curve which is used as a water-line in practice, and the question of the effect of such abrupt change or discontinuity of motion is one which at present can be decided by observation and experiment only. Now it appears from observation and experiment, that the effect of the discontinuity of motion at the bow and stern of a vessel which has an entrance and run of ordinary sharpness and not convex, extends to a very thin layer of water only, and that beyond a short distance from the vessel's side the discontinuity

ceases, through some slight modification of the water-lines of which the mathematical theory is not yet adequate to give an exact account*.

Still, although the effect of the discontinuity in increasing resistance may not yet have been reduced to a mathematical expression, and although it may be so small that our present methods of experimenting have not yet detected it, it must have some value; and it is desirable so to select the limits of the water-line as to make that value as small as possible. In order that the abrupt change of motion may take place in as small a mass of water as possible, it would seem that the limits of the water-line employed in practice should be at or near the point of *slowest gliding*; that is, where the water-line curve is cut by the trajectory of slowest gliding LN, in Plate VIII. fig. 1, and Plate IX. fig. 3, as explained in articles 14, 15, and 16; and that conclusion is borne out by the figures of many vessels remarkable for economy of power.

(26.) *Preferable Figures of Water-Lines.*—In forming a probable opinion as to which, out of all the water-lines generated by a given oval, is to be preferred to the others, regard is to be had to the fact, that every point of maximum disturbance of the level of the water, whether upwards or downwards, that is to say, every point of maximum or minimum speed of gliding (see article 20), forms the origin of a wave, which spreads out obliquely from the vessel (as may easily be observed in smooth water), and so transfers mechanical energy to distant particles of water, which energy is lost. Hence such points should be as few as possible; and the changes of motion at them should be as gradual as possible; and these conditions are fulfilled by the curves described in article 17, by the name of "Lissoneoïds," being those which traverse the point P in the figures, and which may have any proportion of length to breadth, from $\sqrt{3}$ to infinity.

(27.) *Approximate Rules for Construction and Calculation.*—The description of those curves, already given in article 17, has been confined to those properties which are exactly true. The following rules are convenient approximations for practical purposes, *when the proportion of length to breadth is not less than* 4 : 1 (see Plate IX. figs. 3 & 4).

I. A tangent to the curve at Q, the point of slowest gliding, passes very nearly through the point P of greatest breadth.

II. The area PQR enclosed within the water-line is very nearly equal to the rectangle of the breadth PR and excentricity a. (When the length is not less than six times the breadth, this rule is almost perfectly exact.)

* In confirmation of this, experiments made on the steamers 'Admiral' and 'Lancefield,' by Mr. J. R. NAPIER and the author, may be specially referred to. The water-lines of the 'Admiral' are complete trochoids, and tangents to the longitudinal axis at the bow and stern. The engine-power required to drive her at her intended speed was computed from the frictional resistance, according to principles explained in publications already referred to in the note to article 21; and the result of the calculation was closely verified by experiment. The water-lines of the 'Lancefield' are only partly trochoidal, being straight from the point of contrary flexure to the bow, so that, instead of being tangents there to the longitudinal axis, they form with it angles of about $13\frac{1}{2}°$. Yet the same formula which gave the resistance of the 'Admiral' has been found to give also the resistance of the 'Lancefield' without any addition on account of the discontinuity of motion at the bow.

III. For the trajectory of slowest gliding, LN, there may be substituted, without practical error, a straight line cutting the axis OX in L at an angle of 45°; and when this has been done, the excentricity OA or a is almost exactly equal to the length

$$\times \cdot 634 \left(= \frac{3}{3+\sqrt{3}} \right);$$

and this of course is also the ratio of the area to the circumscribed rectangle. The base OL or l also is very nearly equal to (the sum of the length and breadth) $\times \cdot 634$.

IV. Hence the following approximate construction. Given, the common length QR of a set of water-lines of smoothest gliding which are to have a common termination at Q, and their breadths RP_1, RP_2, RP_3, &c.: required, to find their areas, bases, and foci.

Through Q and R draw the straight lines QU and RU, making the angles $RQU=45°$, $QRU=30°$. Through their intersection U draw UV perpendicular to RQ. All the required foci will be in UV; and RV will be the length of the rectangles equivalent to each of the water-line areas; so that

$$\begin{aligned} \text{area } P_1 \, QR_1 &= RV \times RP_1, \\ \text{area } P_2 \, QR_2 &= RV \times RP_2, \\ \&c. \qquad & \qquad \&c. \end{aligned}$$

Through P_1, P_2, P_3, &c. draw lines parallel to RU, cutting QU in L_1, L_2, L_3, &c.: these points will be the ends of the bases required, through which draw the bases $L_1 O_1$, $L_2 O_2$, $L_3 O_3$, &c. parallel to QR, and cutting VU in A_1, A_2, A_3, &c.: these will be the required foci.

The bases and foci and the points P_1, P_2, P_3, &c. being given, the water-lines are to be constructed by the rules given in article 11.

(28.) *Lissoneoïds compared with Trochoids.*—In fig. 5, Plate IX., the full line PQ is a lissoneoïd, and the dotted line Pq a trochoid of the same breadth and area. The curves lie very near together throughout their whole course—the only differences being, that the trochoid is slightly less full and more hollow than the lissoneoïd, but at the same time the trochoid is the longer and has a greater frictional surface. Had the entrance of the trochoid consisted of a straight tangent from its point of contrary flexure (as in the bow of the 'Lancefield,' mentioned in the note to article 25), the two curves would have lain still closer together. The same likeness to a trochoid is found in all lissoneoïds whose length is more than about $3\frac{1}{2}$ times the breadth.

(29.) *Combinations of Bow and Stern.*—Although there is reason to believe that water-lines of equal length and similar form at the bow and stern, such as are produced by using one neoïd curve throughout, are the best on the whole, still the naval architect, should he think fit, can combine two different oögenous neoïds for the bow and stern; or, according to a frequent practice, he may adapt the figure of the stern to motion of the particles in vertical layers instead of horizontal layers; provided he takes care in every case that the midship velocity of gliding (u_0, as given by equation (28 A) of article 13) is the same for each bow water-line and stern water-line at their point of junction.

(30.) *Provisional Formula for Resistance.*—Until the difficulty of integration, mentioned in article 30, shall have been overcome, or until more exact experimental data than we have at present shall have been obtained, the following provisional formula, analogous to that which has been found to agree with the results of experiment on trochoidal and nearly trochoidal lines, as well as some others, may be considered as a probable approximation for lissoneoïds,

$$R=\frac{KWc^2}{2g}\cdot\left(1+4\frac{(u_0-c)^2}{c^2}\right)LG; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where G is the mean girth of the vessel under water; L her total length; u_0 the midship velocity of gliding, found, for a lissoneoïd, by equation (37) of article 17; c the speed of the ship; W the heaviness of water; and K a coefficient of friction (=about ·0036 for a clean surface of paint).

APPENDIX.

Note to Article 11.—The general process of constructing a series of curves whose equation is $\phi(x, y)+\psi(x, y)=$ constant, by drawing lines diagonally through a network consisting of two sets of curves whose equations are respectively $\phi(x, y)=$ constant and $\psi(x, y)=$ constant, is due to Professor CLERK MAXWELL.

SUMMARY OF THE CONTENTS.

Section I.—*Introduction, and Summary of known Principles.*

Section II.—*Properties of Water-Line Curves generated from Ovals, or Oögenous Neoïds.*

FIG. 1.

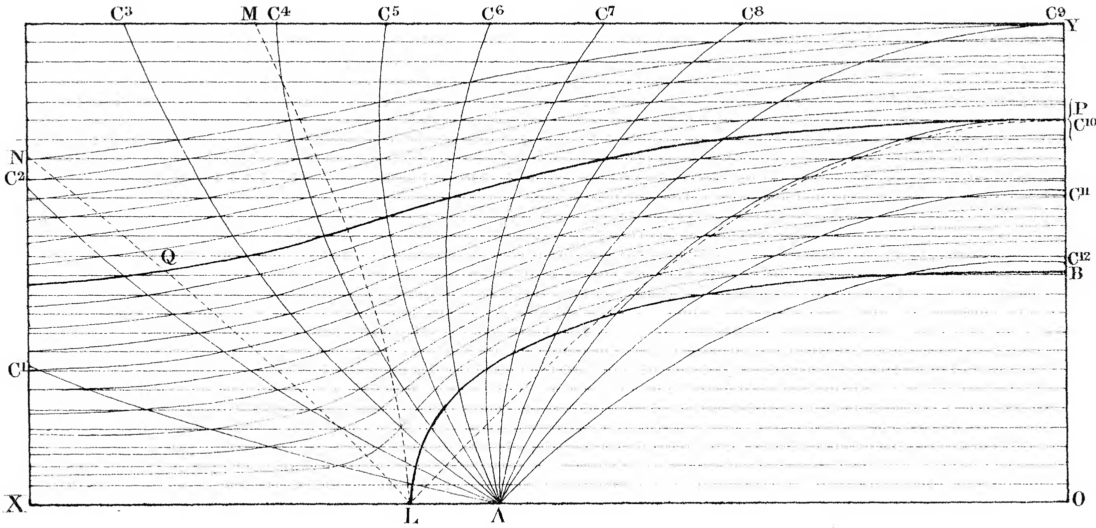

FIG. 2.

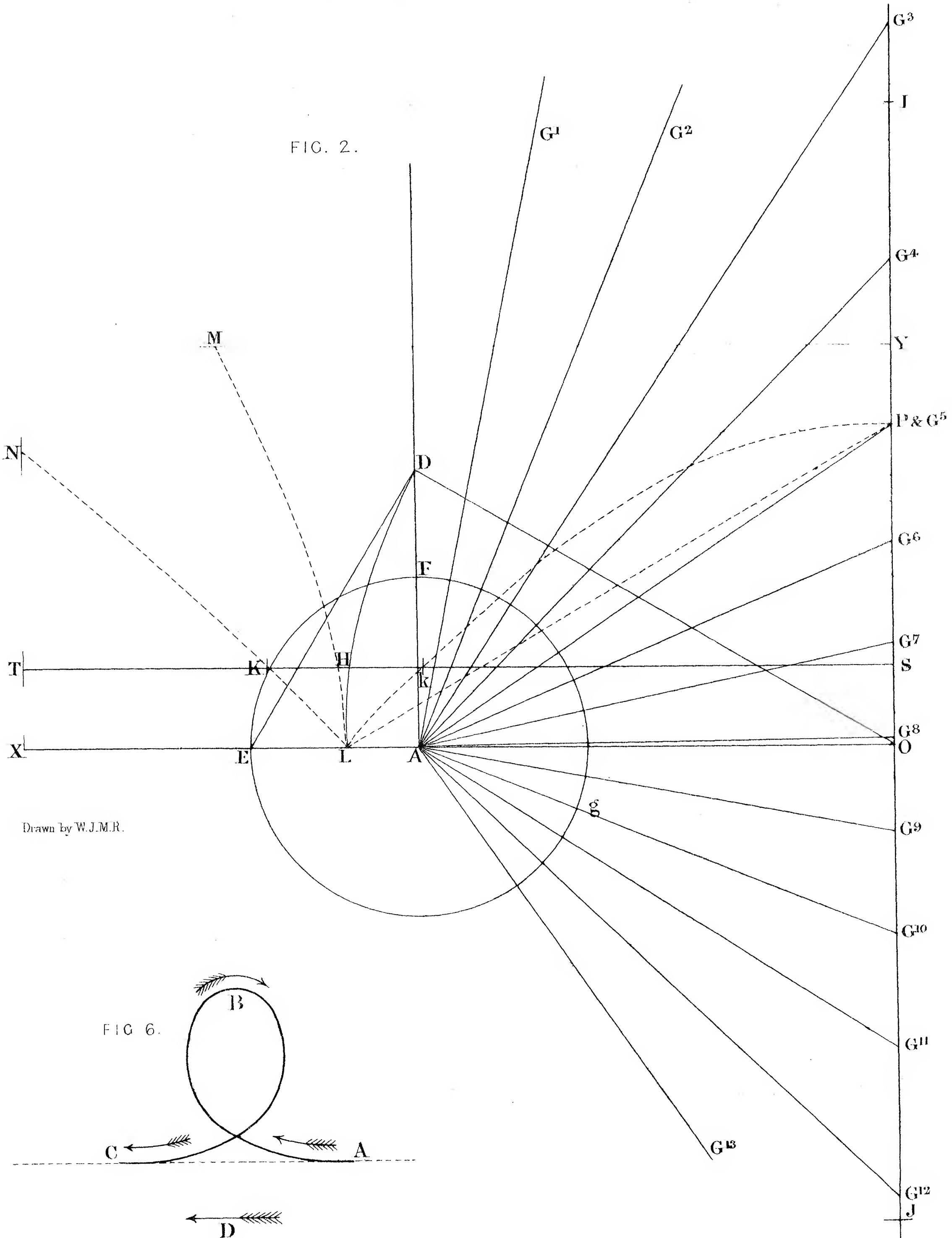

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FIG 6.

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RANKINE ON PLANE WATER-LINES.

FIG. 3.

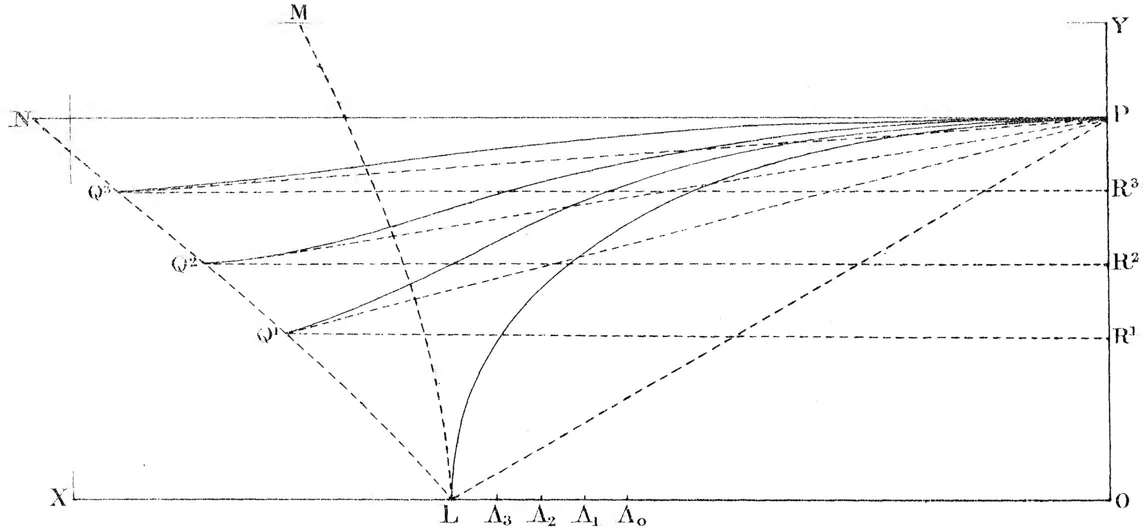


FIG. 4.

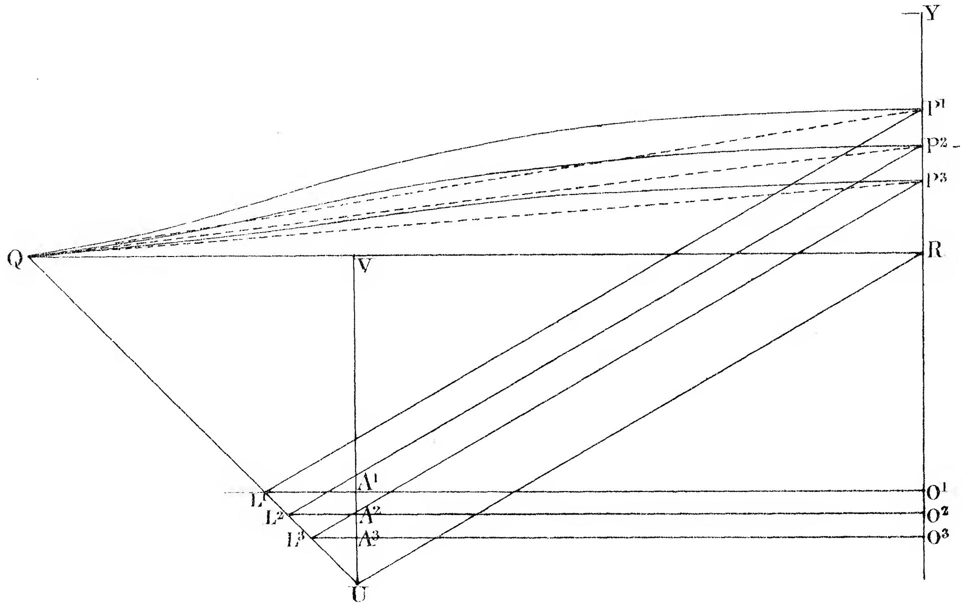
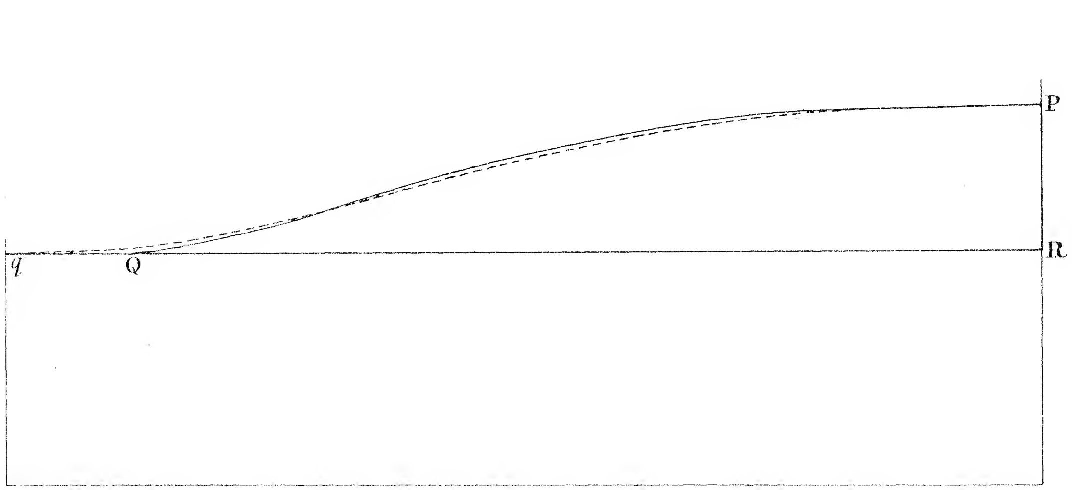


FIG. 5.



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